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Spline Quadrature Formulas

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1. INTRODUCTION

In this paper we develop quadrature formulas for splines with equispaced knots. For special classes of splines, we give simple explicit expressions for the weights in the quadrature formulas in terms of the zeros of the Euler–Frobenius polynomials and show that these weights are positive. The zeros of these polynomials of odd degree up to 15, are given by Nilson [2] and by Schoenberg and Silliman [4] to a high degree of accuracy. The general quadrature formulas can also be used to obtain the cubic natural spline quadrature formula given in Ahlberg, Nilson and Walsh [1, pp. 44–47] and the semicardinal odd order natural spline formula of Schoenberg and Silliman [4].

Two of the quadrature formulas that we derive can be stated as follows. Let S be a spline of odd degree m with knots at the integers $0, 1, \ldots, n$. Suppose that $S^{(2i)}(0^+) = S^{(2i)}(n^-) = 0$, for $2 \leqslant 2i \leqslant m-1$, where $S^{(k)}$ denotes the kth derivative of S. Then

$$\int_0^n S(x)\,dx = (S(0)+S(n))\left(\frac{1}{2}+\frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1}(\lambda_i^n-\lambda_i)/[(\lambda_i^n+1)(\lambda_i-1)]\right)$$

$$+\sum_{j=1}^{n-1} S(j)\left(1-\frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1}(\lambda_i^{n-j}+\lambda_i^j)/(\lambda_i^n+1)\right),$$

where λ_i, $i=1,\ldots,m-1$ with $\lambda_{m-1}<\lambda_{m-2}<\cdots<\lambda_1<0$, are the zeros of the mth Euler–Frobenius polynomial. Moreover, the weights in this quadrature formula are positive.

If S is an integrable semicardinal spline on $[0,\infty)$ with knots at the integers and with $S^{(2i)}(0^+)=0$ for $2 \leqslant 2i \leqslant m-1$, then

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$$\int_0^{\infty} S(x)\,dx = S(0)\left(1/2 + \frac{2}{m+1} \sum_{i=(m+1)/2}^{m-1} 1/(\lambda_i - 1)\right) + \sum_{j=1}^{\infty} S(j)\left(1 - \frac{2}{m+1} \sum_{i=(m+1)/2}^{m-1} \lambda_i^{-j}\right).$$

The weights in this quadrature formula are positive. Both of these formulas do not involve derivative data at the interval end points and their proof in Theorems 2 and 4 below is one of the main objects of this paper. The composite trapezoid formula which is the exact quadrature formula for splines whose odd derivatives vanish at the end points also does not involve derivative data. This is also the case with the semicardinal formula of Schoenberg and Silliman [4]. In the case of the finite interval, we derive in Theorem 9 an explicit quadrature formula for splines whose even derivatives vanish at one end point and whose odd derivatives vanish at the other end point. These explicit quadrature formulas are based on the fact that, for these special classes of splines, the values of the derivatives at the interval end points can be expressed in terms of the values of the spline at the knots and finite powers of the matrix $T = (t_{ij}) = (\binom{j}{i} - \binom{m}{i})$ with $\binom{j}{i} = 0$ if $i > j$. These expressions are given in Theorems 6, 8 and 9. This matrix T plays a basic role in the work of Nilson [2] on spline interpolation. Our quadrature formulas rely on an expression, given in Theorem 7, for the trace of any analytic function of T.

The paper is organized as follows. The necessary definitions and the main results are collected in the next section. The third section contains the proofs of these results. The fourth section is devoted to numerical results and to remarks on extensions.

2. Spline Quadrature Formulas

We consider mth order splines S with knots at the integers $0, 1, \ldots, n$ defined by

(i) $S \in C^{m-1}(0, n)$, that is, S and its derivatives of orders up to $m-1$ are continuous on $(0, n)$;

(ii) S restricted to the interval $[j-1, j]$ is a polynomial of degree $m \geqslant 1$, for $j = 1, 2, \ldots, n$.

Letting B_i be the ith Bernoulli number ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/24$, $B_{2k-1} = 0$ for $k > 1$), we define the $(m-1)$-dimensional column vectors $\mathbf{b}$, $\mathbf{d}$ and $\mathbf{e}$ to have components b_i, d_i and e_i, respectively, given by

$$b_i = -B_{i+1}/(i+1), \qquad d_i = \binom{m}{i}[1 + (-1)^i]/2$$

and

$$e_i = \binom{m}{i} (1 + (-1)^{i+1})/2.$$

We denote the corresponding row vectors by $\mathbf{b}^T$, $\mathbf{d}^T$ and $\mathbf{e}^T$. We define I, J and T to be the $(m-1) \times (m-1)$ matrices with entries given by

$$I = (\delta_{ij}), \qquad J = ((-1)^{i+1} \delta_{ij}), \qquad T = (t_{ij}) = \left(\binom{j}{i} - \binom{m}{i} \right),$$

with $\binom{j}{i} = 0$ if $i > j$ and δ_{ij} being the Kronecker delta.

Let $S^{(j)}(0^+)$ and $S^{(j)}(n^-)$ denote the jth right and left hand derivatives of S at 0 and n, respectively. Let $\mathbf{u}$ and $\mathbf{u}^*$ be the $(m-1)$-dimensional column vectors with respective components

$$u_j = S^{(j)}(0^+)/j! \qquad \text{and} \qquad u_j^* = (-1)^j S^{(j)}(n^-)/j!, \qquad j = 1, 2, \ldots, m-1.$$

We call $\mathbf{u}$ and $\mathbf{u}^*$ the "left end vector" and the "right end vector" of S. With this notation, we can state our first result.

THEOREM 1. *Let S be an mth order spline on $[0, n]$ with knots at the integers $0, 1, \ldots, n$; $n \geqslant 1$. Then the end vectors $\mathbf{u}$ and $\mathbf{u}^*$ of S satisfy the relations*

$$(T^n + J)(\mathbf{u} + \mathbf{u}^*) = -2 \sum_{k=1}^{n-1} S(k)(T^k + T^{n-k})\,\mathbf{e} - 2(S(0) + S(n)) \times (T^n - T)(T - I)^{-1}\,\mathbf{e}, \tag{1}$$

$$(T^n - J)(\mathbf{u} - \mathbf{u}^*) = -2 \sum_{k=1}^{n-1} S(k)(T^k - T^{n-k})\,\mathbf{e} - 2(S(0) + S(n)) \times (T^n + T)(T - I)^{-1}\,\mathbf{e}. \tag{2}$$

The proof of this result will be given in the next section. The integral of S is, of course, given by the Euler–McLaurin formula

$$\int_0^n S(x)\,dx = \sum_{k=1}^{n-1} S(k) + (S(0) + S(n))/2 - \mathbf{b}^T(\mathbf{u} + \mathbf{u}^*). \tag{3}$$

Relations (1) and (2) distinguish an $(m-1)/2$-dimensional hyperplane in which $\mathbf{u} + \mathbf{u}^*$ and $\mathbf{u} - \mathbf{u}^*$ lie. This hyperplane will be explicitly described in Theorem 10 of Section 4. For certain special subspaces of splines, (1) and (2) can be used to calculate $\mathbf{u} + \mathbf{u}^*$ and then (3) yields an explicit quadrature formula. We will proceed to do this for one class of splines after introducing some notation. We note that our proof of (1) and (2) also gives a direct proof of (3).

We follow the terminology of Nilson [2] and define the mth order Hille polynomial by

$$P_m(x) = (1-x)^{m+1}(d/d\log x)^m(1/(1-x)).$$

The mth order Euler–Frobenius polynomial δ_m is then given by

$$\delta_m(x) = P_m(x)/x.$$

From this definition, it readily follows that

$$\delta_1(x) = 1, \qquad \delta_2(x) = 1 + x,$$
$$\delta_m(x) = mx\delta_{m-1}(x) + (1-x)\, d(x\delta_{m-1}(x))/dx.$$

The zeros λ_j of $\delta_m(x)$ are real, negative and distinct [2] and if enumerated in decreasing order $\lambda_{m-1} < \lambda_{m-2} < \lambda_{m-3} < \cdots < \lambda_1 < 0$, then $\lambda_j\lambda_{m-j} = 1$ for $j = 1, 2,..., m-1$ [2, p. 444].

We define a spline S on $[0, n]$ to be "even alternating" if $S^{(j)}(0^+) = S^{(j)}(n^-) = 0$ for j even.

We can now give our next result.

THEOREM 2. *Let S be an even alternating spline of odd order m on $[0, n]$ with equispaced knots at the integers $0, 1,..., n$. Then*

$$\int_0^n S(x)\,dx = (S(0)+S(n))\left(\frac{1}{2} + \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1} (\lambda_i^n - \lambda_i)/[(\lambda_i^n+1)(\lambda_i-1)]\right)$$
$$+ \sum_{j=1}^{n-1} S(j)\left(1 - \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1} (\lambda_i^{n-j} + \lambda_i^j)/(\lambda_i^n+1)\right), \qquad (4)$$

where λ_i, $i = 1,..., m-1$ are the zeros of the mth order Euler–Frobenius polynomial.

When $m = 3$, the even alternating spline reduces to the natural cubic spline. In this case the weights in formula (3) are the same as those given for natural cubic splines in [1, pp. 46–47].

In the next section we shall see that the λ_i appearing in Theorem 2 are the eigenvalues of the matrix T. This fact can be used to obtain the following asymptotic form of Theorem 1.

THEOREM 3. *Let m be an odd integer and let S be an integrable mth order semicardinal spline on* $[0, \infty)$ *with knots at the integers* $0, 1, 2, \ldots$. *Let P_i and Q denote the projections*

$$P_i = \prod_{j \neq i} (T - \lambda_i I)/(\lambda_i - \lambda_j),$$
$$Q = \sum_{|\lambda_i| < 1} P_i. \tag{5}$$

Then

$$\int_0^\infty S(x)\,dx = S(0)/2 + \sum_{k=1}^{\infty} S(k) - \mathbf{b}^T\mathbf{u}, \tag{6}$$

where $\mathbf{u}$ *satisfies the equation*

$$QJ\mathbf{u} = -S(0)\,Q(\mathbf{e} + \mathbf{d}) - 2\sum_{k=1}^{\infty} S(k)\,QT^k\,\mathbf{e}. \tag{7}$$

For even alternating splines the corresponding asymptotic result yields a simple explicit formula which we give in our next result.

THEOREM 4. *Let m be an odd integer and let S be an integrable mth order even alternating semicardinal spline on* $[0, \infty)$ *with knots at the integers* $0, 1, 2, 3, \ldots$. *Then*

$$\int_0^\infty S(x)\,dx = S(0)\left(1/2 + \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1} 1/(\lambda_i - 1)\right)$$
$$+ \sum_{j=1}^{\infty} S(j)\left(1 - \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1} \lambda_i^{-j}\right),$$

where λ_i, $i = 1, 2, \ldots, m-1$, are the zeros of the mth order Euler–Frobenius polynomial.

The above quadrature formulas yield the following result for the weights.

THEOREM 5. *The weights in the quadrature formulas for the even alternating splines in Theorems* 2 *and* 4 *are all positive.*

As seen by the quadrature formula (4) of Theorem 2, Eqs. (1) and (2) of Theorem 1 can be explicitly solved for certain classes of splines and $\mathbf{u}$ and $\mathbf{u}^*$ can be expressed in terms of the values of S at the knots without the use of derivative data. This is indeed the case for even alternating splines for which we get the following results.

THEOREM 6. *Let m be an odd integer and let S be an even alternating spline of order m on $[0, n]$ with knots at the integers $0, 1, \ldots, n$. If $\mathbf{u}$ and $\mathbf{u}^*$ are the end vectors of S, we have*

$$\mathbf{u} = -(T^n - T^{-n})^{-1} \left[2 \sum_{p=1}^{n-1} S(p)(T^{n-p} - T^{p-n})\,\mathbf{e} - S(0)(T^n + T^{-n})\,\mathbf{d} \right.$$
$$\left. + 2S(n)\,\mathbf{d} \right] - S(0)\,\mathbf{e}, \tag{8}$$
$$\mathbf{u}^* = -(T^n - T^{-n})^{-1} \left[2 \sum_{p=1}^{n-1} S(p)(T^{p} - T^{-p})\,\mathbf{e} - S(n)(T^n + T^{-n})\,\mathbf{d} \right.$$
$$\left. + 2S(0)\ \mathbf{d} \right] - S(n)\,\mathbf{e}. \tag{9}$$

The relations (8) and (9) completely determine the even alternating spline of order m which interpolates the points $(j, S(j))\, j = 0, 1, \ldots, n$. They can also be used to obtain the explicit quadrature formula (4) for such splines. We take up the details of the derivations of these results in the next section.

The matrix T plays a basic role in the derivation of our results as well as other questions concerning splines (see the results in [2]). The following result concerning this matrix T is fundamenttal in the proof of Theorems 2 and 4 above, and is of independent interest.

THEOREM 7. *If F is an analytic function in a deleted neighborhood of the origin, then*

$$2(m+1)\,\mathbf{b}^T F(T)\,\mathbf{e} = -\operatorname{Trace} F(T). \tag{10}$$

Alternatively, for the projections P_i defined in (5)

$$2(m+1)\,\mathbf{b}^T P_i \mathbf{e} = -1, \qquad 1 \leqslant i \leqslant m-2. \tag{11}$$

3. DERIVATION OF THE QUADRATURE FORMULA

In this section we give the proofs of the results that were stated above. We start by establishing three lemmas that give some properties of the matrix T. We then give the proof of Theorem 6, which follows easily from these lemmas. The proofs of Theorems 1 and 3, as well as a proof of Eq. (3) based on Theorem 1, are given next. We then proceed to prove Theorem 7, which is a key element in establishing the remainder of our results. The section concludes with the proofs of Theorems 2, 4 and 5.

We start by considering some properties of the $(m-1) \times (m-1)$ matrix T which are essentially already in the literature but which we collect for convenience in Lemma 1, below. We first introduce some notation. Throughout this paper m will be an odd positive integer.

Let S_p be an mth order spline with knots at the integers i, $0 \leqslant i \leqslant n$, and such that $S_p(i) = \delta_{ip}$. We will call S_p a "delta spline" at p. Let $\mathbf{u}_p(i)$ be the column vectors whose jth component is $S_p^{(j)}(i)/j!$, $j = 1, 2, \ldots, m-1$, with $S_p^{(j)}$ denoting the jth derivative of S_p. Let $\mathbf{u}_p^*(i)$ be the column vector whose jth component is $(-1)^j S_p^{(j)}(i)/j!$, $j = 1, 2, \ldots, m-1$. We can now state Lemma 1.

LEMMA 1. *The spectrum of T consists of the $(m-1)$ distinct zeros λ_i, $i = 1, \ldots, m-1$, of the mth Euler–Frobenius polynomial δ_m. So, $\lambda_{m-1} < \lambda_{m-2} < \cdots < \lambda_1 < 0$ and $\lambda_i \lambda_{m-i} = 1$. The eigenvector $v^{(i)}$ corresponding to the eigenvalue λ_i of T has components*

$$v_j^{(i)} = (-1)^{j-1} \binom{m-1}{j-1} (1-\lambda_j)^{j-1} \delta_{m-j}(\lambda_i), \qquad j = 1, 2, \ldots, m-1.$$

Finally, T is similar to its inverse: $T^{-1} = JTJ$ (*trivially*, $J = J^{-1}$) *and*

$$T\mathbf{u}_p(k) = \mathbf{u}_p(k+1), \qquad 0 \leqslant k \leqslant p-2, \qquad p+1 \leqslant k \leqslant n-1, \tag{12}$$

$$T\mathbf{u}_p^*(k+1) = \mathbf{u}_p^*(k), \qquad 0 \leqslant k \leqslant p-2, \qquad p+1 \leqslant k \leqslant n-1. \tag{13}$$

Proof. Since S_p is an mth order spline which vanishes at k for $k \neq p$, then if $0 \leqslant k \leqslant p-2$ or $p+1 \leqslant k \leqslant n-1$, and $x \in [k, k+1]$ we have $S_p(x) = \sum_{j=1}^m [x-k]^j S_p^{(j)}(k)/j!$ and $S_p(k+1) = \sum_{j=1}^m S_p^{(j)}(k)/j! = 0$. The spline matching conditions at the knots are $S_p^{(j)}((k+1)^-) = S_p^{(j)}((k+1)^+)$, yielding the relations

$$\sum_{j=i}^m S_p^{(j)}(k)/(j-i)! = S_p^{(i)}(k+1).$$

Since $S_p^{(m)}(k)/m! = -\sum_{j=1}^{m-1} S_p^{(j)}(k)/j!$, we obtain the relation

$$\sum_{j=1}^{m-1} \left[\binom{j}{i} - \binom{m}{i} \right] S_p^{(j)}(k)/j! = S_p^{(i)}(k+1)/i!, \qquad 1 \leqslant i \leqslant m-1.$$

Recalling the definitions of T, $\mathbf{u}_p$ and $\mathbf{u}_p^*$ we see that this is equivalent to $T\mathbf{u}_p(k) = \mathbf{u}_p(k+1)$, and Eq. (12) holds. A similar argument shows that (13) also holds.

From the definitions of $\mathbf{u}_p$ and $\mathbf{u}_p^*$ we see that $-J\mathbf{u}_p^*(k) = \mathbf{u}_p(k)$. From this and (12) we obtain the relations $TJ\mathbf{u}_p^*(k) = J\mathbf{u}_p^*(k+1)$, and hence, $TJTJ\mathbf{u}_p^*(k) = T\mathbf{u}_p^*(k+1)$, $1 \leqslant k \leqslant p-2$, $p+1 \leqslant k \leqslant n-1$. For $k = 1$, this yields $TJTJ\mathbf{u}_p^*(1) = T\mathbf{u}_p^*(2)$, and using (13) with $k = 1$ we have $TJTJ\mathbf{u}_p^*(1) = \mathbf{u}_p^*(1)$. Since $\mathbf{u}_p^*(1)$ is an arbitrary $(m-1)$ vector we conclude that $T^{-1} = JTJ$. An immediate consequence of this fact is that if λ is an eigenvalue of T then so is $1/\lambda$.

These simple considerations show that the operator T translates the vector $\mathbf{u}_p(k)$ at the knot k to $\mathbf{u}_p(k+1)$ at the knot $k+1$. Nilson [2] considers a related operator S (denoted by T in [2]) with entries s_{ij} given by

$$s_{ij} = m!/[j!(m-i)!] \qquad \text{if} \quad i > j;$$
$$s_{ij} = 1/(j-1)! - m!/[j!(m-1)!] \qquad \text{if} \quad i \leqslant j.$$

If we let Y be the $(m-1) \times (m-1)$ matrix with entries $y_{ij} = j!\,\delta_{ij}$, then it is immediately seen that the (i,j)th entry of $Y^{-1}SY$ is

$$\sum_{l=1}^{m-1} \sum_{p=1}^{m-1} \frac{\delta_{il}}{l!} s_{lp} \delta_{jp}\, p! = \frac{s_{ij}}{i!} j! = \begin{cases} \binom{j}{i} - \binom{m}{i}, & i \leqslant j, \\ -\binom{m}{i}, & i > j. \end{cases}$$

So, $Y^{-1}SY = T$. From this we see that the spectrum of S and that of T are identical. Using Theorem 1 in [2] we obtain our claims concerning the spectrum of T, and the form of the eigenvectors of T. This completes the proof of the lemma.

Since the eigenvalues of T are negative, the matrix $T - I$ is invertible. Letting $\mathbf{r}$ be the vector with components $r_i = (m-i)/[(i+1)(m+1)]$, $i = 1, 2, \ldots, m-1$, we have the following lemma.

LEMMA 2. *Let T, $\mathbf{b}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{r}$ be as defined above. Then*

$$\mathbf{b}^T = \mathbf{r}^T (T-I)^{-1} \qquad \textit{and} \qquad \mathbf{d} = (T+I)(T-I)^{-1}\,\mathbf{e}.$$

Proof. We show that $\mathbf{b}^T(T-I) = \mathbf{r}^T$. Let q_j be the jth component of $\mathbf{b}^T(T-I)$ and note that

$$\begin{aligned} q_j &= \sum_{i=1}^{m-1} -\frac{B_{i+1}}{i+1} \left[\binom{j}{i} - \binom{m}{i} - \delta_{ij} \right] \\ &= \sum_{i=1}^{m-1} \left[\frac{B_{i+1}}{m+1} \binom{m+1}{i+1} - \frac{B_{i+1}}{j+1} \binom{j+1}{i+1} \right] + \frac{B_{j+1}}{j+1} \\ &= \frac{1}{m+1} \left[\sum_{i=0}^{m} B_i \binom{m+1}{i} - B_0 - (m+1) B_1 \right] \\ &\quad - \frac{1}{j+1} \left[\sum_{i=0}^{j} B_i \binom{j+1}{i} - B_0 - (j+1) B_1 + B_{j+1} \right] + \frac{B_{j+1}}{j+1}. \end{aligned}$$

Now using the identity $\sum_{j=0}^{l} B_j \binom{l+1}{j} = 0$ with $l = m-1$ and $j-1$ we get

$$q_j = B_0 \left(\frac{1}{j+1} - \frac{1}{m+1} \right) = (m-j)/[(j+1)(m+1)] = r_j,$$

and the first equation in the lemma is proved. The second equation is a direct consequence of the relation $T(\mathbf{d}-\mathbf{e})=\mathbf{d}+\mathbf{e}$.

We note that

$$\mathbf{b}^T\mathbf{e} = -\sum_{j=1}^{m-1} \frac{B_{j+1}}{j+1}\binom{m}{j} = -\frac{1}{m+1}\sum_{j=2}^{m} B_j \binom{m+1}{j}$$
$$= -\frac{1}{m+1}\left[-B_0-(m+1)B_1\right] = -(m-1)/2(m+1),$$

and we have the relation

$$\mathbf{b}^T\mathbf{e} = -(m-1)/2(2m+1),$$

which will be used below.

Our final lemma is the following.

LEMMA 3. *Let $\mathbf{u}_p$ and $\mathbf{u}_p^*$ be the left and right end vectors of the mth order delta spline S_p. Then*

$$T^p\mathbf{u}_p + JT^{n-p}\mathbf{u}_p^* = -2\mathbf{e}, \qquad p \neq 0, n \tag{14}$$

and

$$\begin{aligned} J\mathbf{u}_0 + T^n\mathbf{u}_0^* &= -\mathbf{e}-\mathbf{d}, \\ T^n\mathbf{u}_n + J\mathbf{u}_n^* &= -\mathbf{e}-\mathbf{d}. \end{aligned} \tag{15}$$

Conversely, any two $(m-1)$ vectors $\mathbf{u}_p$ and $\mathbf{u}_p^*$ satisfying (14) *and* (15) *determine a unique mth order delta spline S_p with end vectors $\mathbf{u}_p$ and $\mathbf{u}_p^*$.*

Proof. For $p \neq 0, n$ let S_p^0 be the mth order polynomial which vanishes at $p-1$ and p and whose jth derivative at p equals $j!$ times the jth component of $T^p\mathbf{u}_p$, $j=1,2,\dots,m-1$. Let S_p^1 be the mth order polynomial which vanishes at p and $p+1$ and whose jth derivative at p equals $(-1)^j j!$ times the jth component of $T^{n-p}\mathbf{u}_p^*$. Then setting

$$S_p(x) = S_p^0(x) + [x-(p-1)]^m, \qquad x \in [p-1,p],$$

and

$$S_p(x) = S_p^1(x) + [p+1-x]^m, \qquad x \in [p,p+1],$$

we need to match derivatives of order $1,2,\dots,m-1$ at p for these two expressions. Performing the differentiations and matching we get the relation

$$j!(T^p\mathbf{u}_p)_j + m!/(m-j)! = (-1)^j j!(T^{n-p}\mathbf{u}_p^*)_j + (-1)^j m!/(m-j)!,$$

hence,

$$(T^p\mathbf{u}_p)_j + (-1)^{j+1}(T^{n-p}\mathbf{u}_p^*)_j = [(-1)^j - 1]\Big/\binom{m}{j}, \qquad j = 1, 2, ..., m-1.$$

In matrix form this is equivalent to

$$T^p\mathbf{u}_p + JT^{n-p}\mathbf{u}_p^* = -2\mathbf{e}.$$

Equation (15) follows from similar considerations at the end points 0 and n.

The converse claim follows from the fact that any solution of (14)–(15) can be used to construct a delta spline S_p with $S_p^{(j)}(i)/j! = T^i\mathbf{u}_p$, $j = 1, ..., m-1$, $S_p(i) = 0$, $i = 0, 1, ..., p-1$, and $(-1)^j S_p^{(j)}(i)/j! = T^{n-i}\mathbf{u}_p^*$, $j = 1, ..., m-1$, $S_p(i) = 0$, $i = p+2, p+3, ..., n$. While for $x \in [p-1, p]$, or $x \in [p, p+1]$, S_p can be constructed as in the first part of the proof of the lemma. This completes the proof. We can now give the proof of Theorem 6.

Proof of Theorem 6. We first show that there exists a unique even alternating delta spline at p, $p \neq 0, n$. If S_p is even alternating and $p \neq 0, n$, $J\mathbf{u}_p^* = \mathbf{u}_p^*$ and (14) becomes $T^p\mathbf{u}_p + T^{p-n}\mathbf{u}_p^* = -2\mathbf{e}$, hence $\mathbf{u}_p + T^{-n}\mathbf{u}_p^* = -2T^{-p}\mathbf{e}$. Multiplying this last equation by J, and using the relations $J\mathbf{u}_p = \mathbf{u}_p$, $JT^{-n} = T^nJ$, $J\mathbf{e} = \mathbf{e}$, we get $\mathbf{u}_p + T^n\mathbf{u}_p^* = -2T^p\mathbf{e}$. Subtracting this last equation from the previous relation involving $\mathbf{u}_p$ and $\mathbf{u}_p^*$ we get $(T^n - T^{-n})\mathbf{u}_p^* = 2(T^{-p} - T^p)\mathbf{e}$. Now, $T^n - T^{-n} = T^{-n}(T^{2n} - I)$ is invertible by the results on the spectrum of T given in Lemma 1. So, $\mathbf{u}_p^*$ is uniquely given by

$$\mathbf{u}_p^* = -2(T^n - T^{-n})^{-1}(T^p - T^{-p})\mathbf{e}. \tag{16}$$

Conversely, any $(m-1)$ vector $\mathbf{u}_p^*$ given by (16) is even alternating, i.e., obeys $J\mathbf{u}_p^* = \mathbf{u}_p^*$. This is easily seen to be the case upon multiplying the right hand side of (16) by J and using $JT^p = T^{-p}J$, $J\mathbf{e} = \mathbf{e}$, $J^{-1} = J$. Similarly, we arrive at the following unique solution of (14)–(15) for the left end vector $\mathbf{u}_p$ of an even alternating S_p:

$$\mathbf{u}_p = -2(T^n - T^{-n})^{-1}(T^{n-p} - T^{p-n})\mathbf{e}, \tag{17}$$

which again is easily seen to obey $J\mathbf{u}_p = \mathbf{u}_p$.

Similarly, we may show, using (15) instead of (14), that there exists a unique even alternating delta spline at each of the end points 0 and n. The resulting end vectors are

$$\mathbf{u}_0 = \mathbf{u}_n^* = -\mathbf{e} + (T^n - T^{-n})^{-1}(T^n + T^{-n})\mathbf{d}, \tag{18}$$

$$\mathbf{u}_0^* = \mathbf{u}_n = -2(T^n - T^{-n})^{-1}\mathbf{d}. \tag{19}$$

Any even alternating spline S may be uniquely represented as a linear combination of the unique even alternating delta splines by

$$S(x) = \sum_{p=0}^{n} S(p)\, S_p(x).$$

The end vectors of S are, hence, given by

$$\mathbf{u} = \sum_{p=0}^{n} S(p)\, \mathbf{u}_p, \qquad \mathbf{u}^* = \sum_{p=0}^{n} S(p)\, \mathbf{u}_p^*.$$

Substituting (16)–(19) into these two equations we get (8) and (9). From the above, these are the only possible values that $\mathbf{u}$ and $\mathbf{u}^*$ can have. This completes proof of Theorem 6.

Proof of Theorem 1. We first show that (1) and (2) hold, then give a derivation of (3). To this end, let S be the spline of Theorem 1, and let S_k be the unique even alternating delta spline at k. Then $S^0 = S - \sum_{k=0}^{n} S(k)\, S_k$ is a null spline with $S^0(j) = 0$ for $j = 0, 1,..., n$. So,

$$S = S^0 + \sum_{k=0}^{n} S(k)\, S_k. \tag{20}$$

From (14) we get $\mathbf{u}_k + T^{-n}J\mathbf{u}_k^* = -2T^{-k}\mathbf{e}$ and $T^{-n}J\mathbf{u}_k + \mathbf{u}_k^* = -2T^{-n+k}\mathbf{e}$. Adding these two equations we have $(I + T^{-n}J)(\mathbf{u}_k + \mathbf{u}_k^*) = -2(T^{-k} + T^{k-n})\,\mathbf{e}$. Multiplying by J we get

$$(T^n + J)(\mathbf{u}_k + \mathbf{u}_k^*) = -2(T^k + T^{n-k})\,\mathbf{e}, \qquad k \neq 0, n. \tag{21}$$

In a similar way we get the relation

$$(T^n - J)(\mathbf{u}_k - \mathbf{u}_k^*) = -2(T^k - T^{n-k})\,\mathbf{e}, \qquad k \neq 0, n. \tag{22}$$

For $k = 0, n$ we use Eq. (15) instead of (14) and Lemma 2 to obtain in an analogous manner

$$\begin{aligned}(T^n + J)(\mathbf{u}_k + \mathbf{u}_k^*) &= -(T^n + I)\,\mathbf{e} + (T^n - I)\,\mathbf{d} \\ &= -(T^n + I)\,\mathbf{e} + (T^n - I)(T + I)(T - I)^{-1}\,\mathbf{e} \\ &= 2(T^n - T)(T - I)^{-1}\,\mathbf{e}, \qquad k = 0, n,\end{aligned} \tag{23}$$

and

$$\begin{aligned}(T^n - J)(\mathbf{u}_k - \mathbf{u}_k^*) &= -(T^n - I)\,\mathbf{e} + (T^n + I)\,\mathbf{d} \\ &= -(T^n - I)\,\mathbf{e} + (T^n + I)(T + I)(T - I)^{-1}\,\mathbf{e} \\ &= 2(T^n + T)(T - I)^{-1}\,\mathbf{e}, \qquad k = 0, n.\end{aligned} \tag{24}$$

For the null spline S_0 with end vectors $\mathbf{u}^0$ and $\mathbf{u}^{0*}$ the matching condition clearly is $T^n\mathbf{u}^0 + J\mathbf{u}^{0*} = 0$, which leads to

$$(J + T^n)(\mathbf{u}^0 + \mathbf{u}^{0*}) = 0 \tag{25}$$

and

$$(J - T^n)(\mathbf{u}^0 - \mathbf{u}^{0*}) = 0. \tag{26}$$

From (20) we see that $\mathbf{u} = \mathbf{u}^0 + \sum_{k=0}^{n} S(k)\,\mathbf{u}_k$ and $\mathbf{u}^* = \mathbf{u}^{0*} + \sum_{k=0}^{n} S(k)\,\mathbf{u}_k^*$. Combining these relations with Eqs. (21)–(26) we obtain Eqs. (1) and (2). Since the mth order Euler–McLaurin formula is exact for mth order splines with equispaced knots, (3) follows, and the proof of Theorem 1 is complete.

We now give an independent proof of (3). From (20) we have

$$\int_0^n S(y)\,dy = \int_0^n S^0(y)\,dy + \sum_{k=0}^{n} S(k) \int_0^n S_k(y)\,dy,$$

and we will obtain Eq. (3) by computing the integrals on the right hand side of this expression. We define $W_k = \int_0^n S_k(y)\,dy$, and note that if $k \neq 0, n$,

$$\begin{aligned} W_k = {} & \sum_{p=0}^{k-1} \int_p^{p+1} \left[\sum_{j=1}^{m} S_k^{(j)}(p)(x-p)^j/j! \right] dx \\ & + \sum_{p=k}^{n-1} \int_p^{p+1} \left[\sum_{j=1}^{m} (-1)^j\, S_k^{(j)}(p)(p-x)^j/j! \right] dx \\ & + \int_{k-1}^{k} (x-(k-1))^m\,dx + \int_k^{k+1} (k+1-x)^m\,dx. \end{aligned}$$

If $k = n$ the above expression holds without the last term, when $k = 0$ it holds without the term before the last (here $\sum_{p=0}^{-1} = \sum_{p=n}^{n-1} \equiv 0$). Performing the integrations and recalling the definition of the vectors $\mathbf{r}$, $\mathbf{u}_p$ and $\mathbf{u}_p^*$, we get for $k \neq 0, n$

$$\begin{aligned} W_k &= 2/(m+1) + \mathbf{r}^T \left[\sum_{p=0}^{k-1} T^p\mathbf{u}_k + \sum_{p=0}^{n-k-1} T^p\mathbf{u}_k^* \right] \\ &= 2/(m+1) + \mathbf{r}^T(T-I)^{-1}[(T^k - I)\,\mathbf{u}_k + (T^{n-k} - I)\,\mathbf{u}_k^*] \\ &= 2/(m+1) + \mathbf{b}^T[(T^k - I)\,\mathbf{u}_k + (T^{n-k} - I)\,\mathbf{u}_k^*], \end{aligned}$$

since, by Lemma 2, $\mathbf{b}^T = \mathbf{r}^T(T-I)^{-1}$.

Now, $J\mathbf{b} = \mathbf{b}^T J = \mathbf{b}^T$, so the expression for W_k, $k \neq 0, n$, can be rewritten as:

$$W_k = 2/(m+1) + \mathbf{b}^T[T^k\mathbf{u}_k + JT^{n-k}\mathbf{u}_k^*] - \mathbf{b}^T[\mathbf{u}_k + \mathbf{u}_k^*].$$

Using Eqs. (13) and (14) we then get

$$W_k = 1 - \mathbf{b}^T(\mathbf{u}_k + \mathbf{u}_k^*), \qquad k \neq 0, n. \tag{27}$$

By symmetry, W_0 and W_n are given by the same expression, and following the steps in the above computation of W_k, we get

$$\begin{aligned} W_n &= 1/(m+1) + \mathbf{r}^T(T-I)^{-1}(T^n - I)\,\mathbf{u}_n \\ &= 1/(m+1) + \mathbf{b}^T[T^n\mathbf{u}_n + J\mathbf{u}_n^*] - \mathbf{b}^T[\mathbf{u}_n + \mathbf{u}_n^*]. \end{aligned}$$

Using (13) and (15) and noting $\mathbf{b}^T\mathbf{d} = 0$, we get

$$W_n = 1/2 - \mathbf{b}^T(\mathbf{u}_n + \mathbf{u}_n^*), \qquad \text{so} \qquad W_0 = 1/2 - \mathbf{b}^T(\mathbf{u}_0 + \mathbf{u}_0^*). \tag{28}$$

For the null spline S^0 we recall that $T^n\mathbf{u}^0 + J\mathbf{u}^{0*} = 0$, and get

$$\begin{aligned} \int_0^n S^0(y)\,dy &= \sum_{p=0}^{n-1} \int_p^{p+1} \left[\sum_{j=1}^{m} S^{0(j)}(p)(y-p)^j/j!\right] dy \\ &= \sum_{p=0}^{n-1} \mathbf{r}^T\mathbf{u}^0(p) = \mathbf{r}^T \sum_{p=1}^{n} T^{p-1}\mathbf{u}^0 \\ &= \mathbf{r}^T(T-I)^{-1}(T^n - I)\,\mathbf{u}^0 = \mathbf{b}^T(T^n - I)\,\mathbf{u}^0 \\ &= \mathbf{b}^T[(T^n\mathbf{u}^0 + J\mathbf{u}^{0*}) - (\mathbf{u}^0 + J\mathbf{u}^{0*})] \\ &= -\mathbf{b}^T(\mathbf{u}^0 + J\mathbf{u}^{0*}) = -\mathbf{b}^T(\mathbf{u}^0 + \mathbf{u}^{0*}). \end{aligned}$$

Collecting this result, (27) and (28) in the expression for the integral, we get

$$\begin{aligned} \int_0^n S(y)\,dy &= [S(0) + S(n)]/2 + \sum_{k=1}^{n-1} S(k) \\ &\quad - \mathbf{b}^T\left[\mathbf{u}^0 + \mathbf{u}^{0*} + \sum_{k=0}^{n} S(k)(\mathbf{u}_k + \mathbf{u}_k^*)\right]. \end{aligned} \tag{29}$$

Noting that $\mathbf{u} = \mathbf{u}^0 + \sum_{k=0}^{n} S(k)\,\mathbf{u}_k$ and $\mathbf{u}^* = \mathbf{u}^{0*} + \sum_{k=0}^{n} S(k)\,\mathbf{u}_k^*$, we obtain Eq. (3). This completes the derivation of (3).

The proof of Theorem 3 proceeds in the same manner as that of Theorem 1. However, the matching conditions (14), (15) and (25) must now reflect the fact that the semicardinal spline is integrable. In particular, this implies that the vectors $\mathbf{u}_p(p+1)$ and $\mathbf{u}^0$ lie in the eigenspace of T corresponding to eigenvalues of modulus less than one, so $(I-Q)\,\mathbf{u}^0 = (I-Q)\,\mathbf{u}_p(p+1) = \mathbf{0}$. Now, since $T\mathbf{x}_i = \lambda_i\mathbf{x}_i$ implies $TJ\mathbf{x}_i = JT^{-1}\mathbf{x}_i = (1/\lambda_i)\,J\mathbf{x}_i$, we see that $(I-Q)\,\mathbf{x} = \mathbf{0}$ is equivalent to $QJ\mathbf{x} = \mathbf{0}$. So, the integrability conditions can be written as, $QJ\mathbf{u}^0 = QJ\mathbf{u}_p(p+1) = \mathbf{0}$. The matching conditions are readily seen to be

$$T^p\mathbf{u}_p - T^{-1}\mathbf{u}_p(p+1) = -2\mathbf{e}, \qquad p \neq 0 \tag{30}$$

$$\mathbf{u}_0 - T^{-1}\mathbf{u}_0(1) = \mathbf{d} - \mathbf{e}. \tag{31}$$

With this, we can now give the proof of Theorem 3.

Proof of Theorem 3. For a semicardinal spline S we have

$$S(x) = S^0(x) + \sum_{k=0}^{\infty} S(k)\, S_k(x),$$

where S^0 is a null semicardinal spline and S_k is a semicardinal delta spline. In particular, the S_k can be taken to be the natural semicardinal fundamental functions (see Lecture 8 in [3]), in which case by the results in the same reference the series in the above equation converges locally uniformly and absolutely, and the integrability of S implies that $\sum_0^\infty S(n)\, S_k$ is integrable (hence, so is S^0), and the integration can be performed termwise. So,

$$\int_0^\infty S(x)\, ds = \int_0^\infty S^0(x)\, dx + \sum_{k=0}^{\infty} S(n) \int_0^\infty S_k(x)\, dx.$$

Now, these integrals can be computed in the same manner as in the proof of Theorem 1, and lead to the expressions:

$$\int_0^\infty S_k(x)\, dx = 2/(m+1) + \mathbf{r}^T \left[\sum_{p=0}^{k-1} T^p\mathbf{u}_k + \sum_{p=0}^{\infty} T^{p-1}\mathbf{u}_k(k+1) \right],$$

the convergence of the infinite series following from the fact that the vector $\mathbf{u}_k(k+1)$ belongs in the eigenspace of T corresponding to eigenvalues with modulus less than one. So, (30) and (31) imply

$$\begin{aligned}
\int_0^\infty S_k(x)\, dx &= 2/(m+1) + \mathbf{r}^T(T-I)^{-1}[(T^k - I)\,\mathbf{u}_k - T^{-1}\mathbf{u}_k(k+1)] \\
&= 2/(m+1) + \mathbf{b}^T[T^k\mathbf{u}_k - T^{-1}\mathbf{u}_k(k+1) - \mathbf{u}_k] \\
&= 2/(m+1) + \mathbf{b}^T \begin{cases} [-2\mathbf{e} - \mathbf{u}_k], & k \neq 0, \\ \mathbf{d} - \mathbf{e} - \mathbf{u}_0, & k = 0, \end{cases} \\
&= \begin{cases} 1 - \mathbf{b}^T\mathbf{u}_k, & k \neq 0, \\ 1/2 - \mathbf{b}^T\mathbf{u}_0, & k = 0. \end{cases}
\end{aligned}$$

Also,

$$\int_0^\infty S^0(x)\, dx = \mathbf{r}^T \sum_{p=0}^{\infty} T^p\mathbf{u}^0 = \mathbf{r}^T(T-I)^{-1}\,\mathbf{u}_0 = -\mathbf{b}^T\mathbf{u}^0.$$

Combining these expressions, we get

$$\int_0^\infty S(x)\,dx = \tfrac{1}{2}S(0) + \sum_{k=1}^{\infty} S(k) - \mathbf{b}^T\left[\mathbf{u}^0 + \sum_{k=0}^{\infty} \mathbf{u}_k S(k)\right].$$

Since $\mathbf{u} = \mathbf{u}^0 + \sum_{k=0}^{\infty} \mathbf{u}_k S(k) = \mathbf{u}^0 + \sum_{k=1}^{\infty} S(k)[T^{k-1}\mathbf{u}_k(k+1) - 2T^{-k}\mathbf{e}] + S(0)[T^{-1}\mathbf{u}_0(1) + \mathbf{d} - \mathbf{e}]$ and using the facts that $QJ\mathbf{u}^0 = QJ\mathbf{u}_p(p+1) = 0$, and $QJT^{-p-1} = T^{p+1}QJ$, we obtain Eqs. (5) and (7) of Theorem 3. This completes the proof of Theorem 3.

The proofs of the rest of our results are based on Theorem 7, whose proof we next present.

Proof of Theorem 7. We first note that (10) and (11) are equivalent. In fact, if (11) holds, then since $T^k = \sum_{i=1}^{m-1} \lambda_i^k p_i$, k an integer, then $\mathbf{b}^T T^k \mathbf{e} = -\sum_{i=1}^{m-1} \lambda_i^k / 2(m+1)$, and (10) follows since $F(T)$ has a convergent expansion in powers of T. Conversely, let (10) hold, then $-2(m+1)\,\mathbf{b}^T T^k E = \sum_{i=1}^{m-1} -2(m+1)\,\mathbf{b}^T P_i \mathbf{e} \lambda_i^k = \sum_{i=1}^{m-1} \lambda_i^k = \text{Trace } T^k$, holds for all integers k. Now, the eigenvalues λ_i are all distinct, so, since $\lambda_1 = 1/\lambda_{m-1}$, multiplying by $\lambda_1^k(|\lambda_1| < |\lambda_i|$ for $i \neq 1)$ and letting $k \to \infty$, we get $-2(m+1)\,\mathbf{b}^T P_1 \mathbf{e} = \lim_{k\to\infty} \sum_{i=1}^{m-1} -2(m+1)\,\mathbf{b}^T P_i \mathbf{e}(\lambda_i \lambda_1)^k = \lim_{k\to\infty} \sum_{i=1}^{m-1} (\lambda_i/\lambda_{m-1})^k = 1$. Repeating this with $\lambda_2, \ldots, \lambda_{m-1}$, we see that (11) holds.

We now introduce some notation that will be used in the rest of the proof. Let $\mathbf{z}$ and $\mathbf{1}$ be $(m-1)$ column vectors with components

$$z_i = \binom{m}{i-1}, \qquad 1_i = 1,$$

and let $\mathbf{h} = \mathbf{e} + \mathbf{d}$, so $h_i = \binom{m}{i}$. Let A and B be the $(m-1) \times (m-1)$ matrices with entries $a_{ij} = \binom{j}{i}$ and $b_{ij} = -\binom{m}{i}$, with $\binom{j}{i} = 0$, if $i > j$. Clearly, $T = A + B$, and we state four identities which will be used in completing our proof. These are

$$2(m+1)\,\mathbf{b}^T T^n \mathbf{e} = (m+1)\,\mathbf{r}^T T^{n-1}\mathbf{h} = \mathbf{1}^T T^{n-1}\mathbf{z}, \tag{32}$$

$$\begin{aligned}\mathbf{1}^T A^n \mathbf{z} = {} & -(n+1)\,\text{Trace}(A^n B) - \text{Trace}(A^{n+1}) \\ & + \sum_{i=0}^{n-1} \text{Trace}(A^{n-i})\,\text{Trace}(A^i B),\end{aligned} \tag{33}$$

$$\begin{aligned}\text{Trace } T^{n+1} = {} & (n+1)\,\text{Trace}(A^n B) + \text{Trace}(A^{n+1}) \\ & + \sum_{i=0}^{n-1} \text{Trace}(T^{n-i} - A^{n-i})\,\text{Trace}(A^i B)\end{aligned} \tag{34}$$

and

$$\text{Trace}(A^i B)(\mathbf{1}^T T^l \mathbf{z}) = \mathbf{1}^T A^i B T^l \mathbf{z}. \tag{35}$$

Before giving the proof of (32)–(35) we show how they lead to the proof of the theorem. From (32) it is seen that all we need to show is that

$$-\text{Trace } T^n = \mathbf{1}^T T^{n-1}\mathbf{z} \qquad \text{for all } n = 1, 2,\ldots \tag{36}$$

(recall that from (10), $2(m+1)\,\mathbf{b}^T\mathbf{e} = -(m-1) = -\text{Trace } T^0$), since (36) implies that $2(m+1)\,\mathbf{b}^T T^n\mathbf{e} = -\text{Trace } T^n$ for all such n. But, from (36), $2(m+1)\,\mathbf{b}^T T^{-n}\mathbf{e} = 2(m+1)\,\mathbf{b}^T J T^n J\mathbf{e} = 2(m+1)\,\mathbf{b}^T T^n\mathbf{e} = -\text{Trace } T^n = -\text{Trace } T^{-n}$, and (36) implies the same relation holds for $n = -1, -2, \ldots$. In turn these relations imply that (10), hence, (11) holds. We now prove (36) by induction.

For $n = 1$, we have to show that $-\text{Trace } T = \mathbf{1}^T\mathbf{z}$. Now, $\mathbf{1}^T\mathbf{z} = \sum_{i=1}^{m-1}\binom{m}{i-1} = \sum_{i=0}^{m-2}\binom{m}{i} = 2^m - m - 1$. Also,

$$-\text{Trace } T = \sum_{i=1}^{m-1}\left[\binom{m}{i} - \binom{i}{i}\right] = \sum_{i=1}^{m-1}\binom{m}{i} - m + 1 = 2^m - m - 1,$$

and (36) holds for $n = 1$. Assume that it holds for $i \leqslant n$. Then from (34) we get

$$\begin{aligned}-\text{Trace } T^{n+1} = {} & -(n+1)\,\text{Trace } A^n B - \text{Trace } A^{n+1} \\ & + \sum_{i=0}^{n-1} (\text{Trace } A^{n-i})(\text{Trace } A^i B) \\ & - \sum_{i=0}^{n-1} (\text{Trace } T^{n-i})\,\text{Trace } A^i B,\end{aligned}$$

and using (33) and the induction hypothesis, this yields

$$-\text{Trace } T^{n+1} = \mathbf{1}^T A^n\mathbf{z} + \sum_{i=0}^{n-1} (\mathbf{1}^T T^{n-i-1}\mathbf{z})\,\text{Trace } A^i B.$$

Using (35) we now get

$$\begin{aligned}-\text{Trace } T^{n+1} &= \mathbf{1}^T\left[A^n + \sum_{i=0}^{n-1} A^i B T^{n-i-1}\right]\mathbf{z} \\ &= \mathbf{1}^T\left[A^n + A^{n-1}B + \sum_{i=0}^{n-2} A^i B T^{n-i-1}\right]\mathbf{z} \\ &= \mathbf{1}^T\left[A^{n-1}T + A^{n-2}BT + \sum_{i=0}^{n-3} A^i B T^{n-i-1}\right]\mathbf{z} \\ &= \mathbf{1}^T\left[A^{n-2}T^2 + A^{n-3}BT^2 + \sum_{i=0}^{n-4} A^i B T^{n-i-1}\right]\mathbf{z} \\ &= \mathbf{1}^T T^n\mathbf{z}.\end{aligned}$$

This completes the induction and the proof of (36) for $n = 1, 2, \ldots$. From what was already established, the proof of Theorem 7 will be completed once relations (32)–(35) are proved.

We start by proving the first equality in (32). Recall that from Lemma 2, $\mathbf{b}^T(T-I) = \mathbf{r}^T$, and hence,

$$\mathbf{b}^T T^n \mathbf{e} = \mathbf{b}^T (T-I)\, T^n (T-I)^{-1}\, \mathbf{e} = \mathbf{r}^T T^{n-1} T(T-I)^{-1}\, \mathbf{e}.$$

So, we need to show that $2T(T-I)^{-1}\,\mathbf{e} = \mathbf{h}$, or else, $2\mathbf{e} = (T-I)^{-1}\,\mathbf{h}$. In fact,

$$2T(T-I)^{-1}\,\mathbf{e} = (T+I)(T-I)^{-1}\,\mathbf{e} + (T-I)(T-I)^{-1}\,\mathbf{e} = \mathbf{d} + \mathbf{e} = \mathbf{h}.$$

This establishes the first equality in (32).

In order to prove the second equality (32) we start by noting that

$$\begin{aligned} t_{ij} h_j &= \left[\binom{j}{i} - \binom{m}{i}\right]\binom{m}{j} = \binom{j}{i}\binom{m}{m-j} - \binom{m}{i}\binom{m}{m-j} \\ &= \binom{m}{i}\left[\binom{m-i}{m-j} - \binom{m}{m-j}\right] = h_i t_{m-j,m-i}. \end{aligned}$$

Using this relation, we get

$$\begin{aligned} (m+1)\,\mathbf{r}^T T^{n-1}\mathbf{h} &= \sum_{s_1=1}^{m-1} \sum_{s_2=1}^{m-1} \cdots \sum_{s_n=1}^{m-1} (m+1)\, r_{s_1} t_{s_1 s_2} t_{s_2 s_3} \cdots t_{s_{n-1} s_n} h_{s_n} \\ &= \sum_{s_1=1}^{m-1} \cdots \sum_{s_n=1}^{m-1} (m+1)\, r_{s_1} h_{s_1} t_{m-s_2, m-s_1} \cdots t_{m-s_n, m-s_{n-1}} \\ &= \sum_{u_1=1}^{m-1} \cdots \sum_{u_n=1}^{m-1} (m+1)\, r_{m-u_n} h_{m-u_n} t_{u_{n-1}, u_n} \cdots t_{u_1, u_2}. \end{aligned}$$

Noting that

$$\begin{aligned} (m+1)\, r_{m-u_n} h_{m-u_n} &= (m+1) \frac{u_n}{(m-u_n+1)(m+1)} \binom{m}{m-u_n} \\ &= \binom{m}{u_n - 1} = z_{u_n}, \end{aligned}$$

we get

$$(m+1)\,\mathbf{r}^T T^{n-1}\mathbf{h} = \sum_{u_1=1}^{m-1} \cdots \sum_{u_n=1}^{m-1} t_{u_1, u_2} \cdots t_{u_{n-1}, u_n} z_{u_n} = \mathbf{1}^T T^{n-1} \mathbf{z},$$

which is the second equation in (32)

We next note that, if $G=(g_{ij})$ is an $(m-1)\times(m-1)$ matrix, then $\text{Trace}(GB)=-\sum_{i,j=1}^{m-1} g_{ij}\binom{m}{j}=-\mathbf{1}^T G\mathbf{h}$. Also, if $H=(h_{ij})$, is an $(m-1)\times(m-1)$ matrix, then $(\text{Trace}(GB))\,\mathbf{1}^T H\mathbf{z}=(-\sum_{i,j} g_{ij}\binom{m}{j})(\sum_{k,l} h_{kl}\binom{m}{l-1})=-\sum_{i,j,k,l} g_{ij}\binom{m}{j} h_{kl}\binom{m}{l-1}=\mathbf{1}^T GBH\mathbf{z}$. In particular, letting $G=A^i$ and $H=T^j$, we get Eq. (35). Finally, we note that

$$\begin{aligned}\text{Trace}(GBHB) &= \sum_{i,j,k,l} g_{ij}\binom{m}{j} h_{kl}\binom{m}{l}=\left(\sum_{i,j} g_{ij}\binom{m}{j}\right)\left(\sum_{k,l} h_{kl}\binom{m}{l}\right)\\ &= (\text{Trace}\,GB)(\text{Trace}\,HB).\end{aligned}$$

We now give the proof of (33). We see from the above that $\text{Trace}(A^nB)=-\mathbf{1}^TA^n\mathbf{h}=-\sum_{i=1}^{m-1}[(n+1)^i-n^i]\binom{m}{i}$, since the ith entry in $\mathbf{1}^TA^n$ is $(n+1)^i-n^i$, which can be seen from the following simple inductive argument. For $n=0$, $\mathbf{1}^TA^0=\mathbf{1}^T$, and the ith entry is one, satisfying the above relation. Assuming that the relation holds for $(n-1)$, we have

$$\begin{aligned}(\mathbf{1}^TA^n)_i=(\mathbf{1}^TA^{n-1}A)_i &= \sum_{k=1}^{m-1}[n^k-(n-1)^k]\binom{i}{k}\\ &= \sum_{k=0}^{i}[n^k-(n-1)^k]\binom{i}{k}\\ &= (n+1)^i-n^i,\end{aligned}$$

so the formula holds for all $n\geqslant 0$. Hence,

$$\begin{aligned}\text{Trace}(A^nB) &= \sum_{i=0}^{m}[(n+1)^i-n^i]\binom{m}{i}+(n+1)^m-n^m\\ &= -(n+2)^m+2(n+1)^m-n^m.\end{aligned}$$

Next, we note that $\text{Trace}\,A^n=\sum_{i=1}^{m-1}\binom{i}{i}^n=m-1$, since A is tridiagonal with diagonal entries $\binom{i}{i}=1$. So,

$$\begin{aligned}&-(n+1)\,\text{Trace}(A^nB)-\text{Trace}(A^{n+1})+\sum_{i=0}^{n-1}\text{Trace}(A^{n-i})\,\text{Trace}(A^iB)\\ &\quad=(n+1)[(n+2)^m-2(n+1)^m+n^m]-(m-1)\\ &\qquad-\sum_{i=0}^{n-1}(m-1)[(i+2)^m-2(i+1)^m+i^m]\\ &\quad=(n+1)[(n+2)^m-2(n+1)^m+n^m]\\ &\qquad-(m-1)\left[1+\sum_{i=2}^{n+1}i^m-2\sum_{i=1}^{n}i^m+\sum_{i=0}^{n-1}i^m\right]\\ &\quad=(n+1)[(n+2)^m-2(n+1)^m+n^m]-(m-1)[(n+1)^m-n^m].\end{aligned}$$

We will now show that this last expression is also equal to $\mathbf{1}^T A^n \mathbf{z}$, and hence, that (33) holds. This is again a simple computation:

$$\begin{aligned}
\mathbf{1}^T A^n \mathbf{z} &= \sum_{i=1}^{m-1} [(n+1)^i - n^i] \binom{m}{i-1} \\
&= (n+1) \left[\sum_{i=0}^{m} (n+1)^i \binom{m}{i} - m(n+1)^{m-1} - (n+1)^m \right] \\
&\quad - n \left[\sum_{i=0}^{m} n^i \binom{m}{i} - mn^{m-1} - n^m \right] \\
&= (n+1)[(n+2)^m - m(n+1)^{m-1} - (n+1)^m] \\
&\quad - n[(n+1)^m - mn^{m-1} - n^m] \\
&= (n+1)[(n+2)^m - 2(n+1)^m + n^m] \\
&\quad - (m-1)[(n+1)^m - n^m],
\end{aligned}$$

which is the desired equality. This proves that (33) holds.

We finally come to the proof of (34). We first show that

$$T^{n+1} = \sum_{i=0}^{n} A^i B A^{n-i} + A^{n+1} + \sum_{i=0}^{n-1} (T^{n-i} - A^{n-i}) B A^i, \quad n = 0, 1, 2, \ldots. \tag{37}$$

It is clear that (37) holds for $n = 0$ and $n = 1$, each of the sides of the equation reducing to $A + B$ if $n = 0$, and to $A^2 + B^2 + AB + BA$ if $n = 1$. Assuming that (37) holds for $n = k$, we see that

$$\begin{aligned}
T^{(k+1)+1} &= (A+B)\, T^{k+1} = (A+B) \left[\sum_{i=0}^{k} A^i B A^{k-i} + A^{k+1} \right. \\
&\quad \left. + \sum_{i=0}^{k-1} (T^{k-i} - A^{k-i}) B A^i \right] \\
&= \sum_{i=0}^{k} A^{i+1} B A^{(k+1)-(i+1)} + \sum_{i=0}^{k} B A^i B A^{k-i} + A^{(k+1)+1} \\
&\quad + B A^{k+1} + \sum_{i=0}^{k-1} (T^{(k+1)-i} - A^{(k+1)-i}) B A^i - \sum_{i=0}^{k-1} B A^{k-i} B A^i \\
&= \sum_{i=0}^{k+1} A^i B A^{(k+1)-i} - B A^{k+1} + B^2 A^k + A^{(k+1)+1} + B A^{k+1} \\
&\quad + \sum_{i=0}^{k} (T^{(k+1)-i} - A^{(k+1)-i}) B A^i - (T - A) B A^k \\
&= \sum_{i=0}^{k+1} A^i B A^{(k+1)-i} + A^{(k+1)+1} + \sum_{i=0}^{k} (T^{(k+1)-i} - A^{(k+1)-i}) B A^i.
\end{aligned}$$

So, (37) holds for $n = k + 1$, and hence, for all $n = 0, 1, 2,\ldots$. We now obtain Eq. (34) by taking the trace of both sides of Eq. (37). We note that $\operatorname{Trace}(A^iBA^{n-i}) = \operatorname{Trace}(A^nB)$, and that

$$\sum_{i=0}^{n-1} \operatorname{Trace}[(T^{n-i} - A^{n-i})\, BA^i] = \sum_{i=0}^{n-1} \operatorname{Trace}(T^{n-i} - A^{n-i}) \operatorname{Trace}(A^iB). \tag{38}$$

The first relation follows from the fact that $\operatorname{Trace}(CD) = \operatorname{Trace}(DC)$ for all square matrices C and D, while the second can be obtained by writing the terms of $(T^{n-i} - A^{n-i})\, BA^i$ in the form $(F_j(A, B)\, BA^j)(BA^i)$, where $F_j(A, B)$ is a product involving only matrices A and B and is of total order $n - i - j$. Then $\operatorname{Trace}[(F_j(A, B)\, BA^j)(BA^i)] = \operatorname{Trace}(A^iF_j(A, B)\, B) \operatorname{Trace}(A^jB) = \operatorname{Trace}(F_j(A, B)\, BA^i) \operatorname{Trace}(A^jB)$, which follows from the relations $\operatorname{Trace}(CD) = \operatorname{Trace}(DC)$ and $\operatorname{Trace}(GBHB) = \operatorname{Trace}(GB) \operatorname{Trace}(HB)$, which was already established. Using this, it is seen that both sides of (38) are sums of terms of the form $\operatorname{Trace}(F_j(A, B)\, BA^i) \operatorname{Trace}(A^jB)$, and that there is a one to one onto correspondence between these terms on both sides of (38). So, (38) holds, and using it on taking the trace of both sides of (37) we get Eq. (34).

Equations (32)–(35) have now been established, and hence, the proof of Theorem 7 is completed.

We now give the proofs of Theorems 2, 4 and 5.

Proof of Theorem 2. From Theorem 6, we know that if S is an even alternating spline it has the unique representation $S = \sum_{k=0}^{n} S(k)\, S_k$, where S_k is the even alternating spline satisfying $S_k(j) = \delta_{kj}$. From (21)

$$(J + T^n)(\mathbf{u}_k + \mathbf{u}_k^*) = -2(T^k + T^{n-k})\, \mathbf{e}, \qquad k \neq 0, n.$$

Hence, since S_k is even alternating $(I + T^n)(\mathbf{u}_k + \mathbf{u}_k^*) = -2(T^k + T^{n-k})\, \mathbf{e}$, and using the invertibility of $(I + T^n)$ and Eq. (10) of Theorem 7, we now get

$$\begin{aligned} \mathbf{b}^T(\mathbf{u}_k + \mathbf{u}_k^*) &= -2\mathbf{b}^T(I + T^n)^{-1}(T^k + T^{n-k})\, \mathbf{e} = \frac{1}{m+1} \sum_{i=1}^{m-1} \frac{\lambda_i^k + \lambda_i^{n-k}}{1 + \lambda_i^n} \\ &= \frac{2}{m+1} \sum_{i=(m+1)/2}^{m-1} \frac{\lambda_i^{n-k} + \lambda_i^k}{1 + \lambda_i^n}, \qquad k \neq 0, n, \end{aligned}$$

since $\lambda_i \lambda_{m-i} = 1$. So, using Eq. (3), we see that we have computed the weights W_k, $k \neq 0, n$, in the quadrature formula for even alternating splines to be

$$W_k = 1 - \frac{2}{m+1} \sum_{i=(m+1)/2}^{m-1} \frac{\lambda_i^{n-k} + \lambda_i^k}{1 + \lambda_i^n}, \qquad k \neq 0, n.$$

The weights W_0 and W_n, can now be computed by noting that $S(x)=1$ is an even alternating spline, hence

$$\int_0^n dx = n = \sum_{k=0}^{n} W_k = W_0 + W_n + \sum_{k=1}^{n-1}\left[1 - \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1}\frac{\lambda_i^{n-k}+\lambda_i^k}{1+\lambda_i^n}\right].$$

So,

$$\begin{aligned} W_0 + W_n &= n-(n-1) + \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1}\frac{1}{1+\lambda_i^n}\sum_{k=1}^{n-1}(\lambda_i^{n-k}+\lambda_i^k) \\ &= 1 + \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1}\frac{2}{1+\lambda_i^n}\left[\frac{\lambda_i-\lambda_i^n}{1-\lambda_i}\right] \\ &= 1 + \frac{4}{m+1}\sum_{i=(m+1)/2}^{m-1}\frac{\lambda_i^n-\lambda_i}{(1+\lambda_i^n)(\lambda_i-1)}. \end{aligned}$$

By symmetry, $W_0 = W_n$, so each is given by

$$\frac{1}{2} + \frac{2}{m+1}\sum_{i=(m+1)/2}^{m-1}\frac{\lambda_i^n-\lambda_i}{(1+\lambda_i^n)(\lambda_i-1)}.$$

This completes the proof of Theorem 2.

Proof of Theorem 4. Since S is even alternating and integrable, $\int_0^\infty S(x)\,dx = \lim_{b\to\infty}\int_0^b S(x)\,dx$, and $S(b)\to 0$ as $b\to\infty$. So, using equation (3) to evaluate $\int_0^b S(x)\,dx$ and noting that $|\lambda_i|>1$ for $(m+1)/2 \leqslant i \leqslant m-1$, the formula of Theorem 4 follows directly by taking the limit as $b\to\infty$. This completes the proof of Theorem 4.

Proof of Theorem 5. The result is immediate for the asymptotic form of the weights in Theorem 4. For, since $\lambda_i < -1$ when $i=(m+1)/2$, $(m+3)/2,\ldots, m-1$, $|\lambda_i^{-j}|<1$, $1\leqslant j<\infty$, and $|1/(\lambda_i-1)|<1/2$, implying that $1/2 + 2/(m+1)\sum_{i=(m+1)/2}^{m-1} 1/(\lambda_i-1) > 0$, and $1-2/(m+1) \times \sum_{i=(m+1)/2}^{m-1}\lambda_i^{-j} > 0$. So, the weights in the formula of Theorem 4 are all positive.

The proof for the weights given by formula (4) of Theorem 2 is only slightly more involved. First, if $1\leqslant j\leqslant n-1$, we note that

$$\begin{aligned} |\lambda_i^{n-j}+\lambda_i^j|/|\lambda_i^n+1| &= |(-1)^n|\lambda_i|^{n-j}+|\lambda_i|^j|/|(-1)^n|\lambda_i|^n+1| \\ &= ||\lambda_i|^{-n/2+j}+(-1)^n|\lambda_i|^{n/2-j}|/||\lambda_i|^{-n/2}+(-1)^n|\lambda_i|^{n/2}| < 1, \end{aligned}$$

since the functions $|\lambda_i|^x+|\lambda_i|^{-x}$ and $|\lambda_i|^x-|\lambda_i|^{-x}$ are monotone increasing for $|\lambda_i|>1$, and $|n/2-j|\leqslant n/2-1<n/2$. This implies that $1-2/(m+1)$

$\times \sum_{i=(m+1)/2}^{m-1} (\lambda_i^{n-j} - \lambda_i^j)/(\lambda_i^n + 1) > 0$, for $1 \leqslant j \leqslant n-1$. For $j = 0, n$, and n odd, we have $|\lambda_i^n - \lambda_i| < |\lambda_i^n + 1|$, and since $|1/(\lambda_i - 1)| < 1/2$ we have

$$\frac{1}{2} + \frac{2}{m+1} \sum_{(m+1)/2}^{m-1} (\lambda_i^n - \lambda_i)/[(\lambda_i^n + 1)(\lambda_i - 1)]$$
$$> 1/2 - \frac{2}{m+1} \sum_{i=(m+1)/2}^{m-1} (1/2) \geqslant 0.$$

If n is even, we note that

$$(\lambda_i^n - \lambda_i)/[(\lambda_i^n + 1)(\lambda_i - 1)] = (\lambda_i^n - \lambda_i)/(\lambda_i^{n+1} - 1 - \lambda_i^n + \lambda_i).$$

Now, since $\lambda_i < -1$, $|\lambda_i^n - \lambda_i| < |\lambda_i^{n+1} - 1|$ follows from the fact that the polynomial function $x^{n+1} + 1 - (x^n + x)$ is positive for $x > 1$; and $|\lambda_i^{n+1} - 1 - \lambda_i^n + \lambda_i| = |\lambda_i^{n+1} - 1| + |\lambda_i^n - \lambda_i| > 2\,|\lambda_i^n - \lambda_i|$. So,

$$\frac{1}{2} + \frac{2}{m+1} \sum_{i=(m+1)/2}^{m-1} (\lambda_i^n - \lambda_i)/[(\lambda_i^n + 1)(\lambda_i - 1)]$$
$$> \frac{1}{2} - \frac{2}{m+1} \sum_{i=(m+1)/2}^{m-1} \frac{\lambda_i^n - \lambda_i}{2\,|\lambda_i^n - \lambda_i|} \geqslant 0.$$

Hence, the weights are positive in this case also and the proof of Theorem 5 is completed.

With this we have completed the proofs of all the results that were presented in Section 2. In the next section we will give some numerical results based on our theorems and some discussion of related results.

4. Numerical and Related Results

This section is devoted to the presentation of some sample computations of the explicit quadrature weights that we derived for even alternating splines as well as of some related results. We start by looking at an obvious counterpart of the even alternating splines. We define an mth order spline S to be "odd alternating" if $S^{(2j-1)}(0^+) = S^{(2j-1)}(n^-) = 0$, for $1 \leqslant j \leqslant (m-1)/2$, $m > 2$ an odd integer. We have the following analogue of Theorem 6 whose proof is omitted since it is similar to that of Theorem 6.

Theorem 8. *Under the same conditions as in Theorem 6, except that S is assumed to be odd alternating instead of even alternating, the end vectors $\mathbf{u}$ and $\mathbf{u}^*$ of S satisfy*

$$\mathbf{u} = -(T^n - T^{-n})^{-1} \left[2 \sum_{p=1}^{n-1} S(p)(T^{n-p} + T^{p-n})\,\mathbf{e} - 2S(0)\,\mathbf{e} \right.$$

$$\left. - S(n)(T^n + T^{-n})\,\mathbf{e} \right] - S(0)\,\mathbf{d},$$

$$\mathbf{u}^* = -(T^n - T^{-n})^{-1} \left[2 \sum_{p=1}^{n-1} S(p)(T^p + T^{-p})\,\mathbf{e} - S(0)(T^n + T^{-n})\,\mathbf{e} \right.$$

$$\left. - 2S(n)\,\mathbf{e} \right] - S(n)\,\mathbf{d}.$$

From the Euler–McLaurin formula it is clear that the trapezoid rule is an exact quadrature formula for odd alternating splines.

We now look at the mixed case of "even–odd splines," which are defined by the restrictions $S^{(2j)}(0^+) = S^{(2j-1)}(n^-) = 0$. The formula for the end vectors of such splines are given in the following theorem.

THEOREM 9. *Under the same conditions as Theorem* 6, *except that S is assumed to be even–odd alternating instead of even alternating, the end vectors* $\mathbf{u}$ *and* $\mathbf{u}^*$ *of S satisfy*

$$\mathbf{u} = -(T^n + T^{-n})^{-1} \left[2 \sum_{k=1}^{n-1} S(k)(T^{n-k} + T^{k-n})\,\mathbf{e} \right.$$

$$\left. - S(0)(T^n - T^{-n})\,\mathbf{d} + 2S(n)\,\mathbf{e} \right] - S(0)\,\mathbf{e}, \tag{39}$$

$$\mathbf{u}^* = -(T^n + T^{-n})^{-1} \left[2 \sum_{k=1}^{n-1} S(k)(T^k + T^{-k})\,\mathbf{e} \right.$$

$$\left. + 2S(0)\,\mathbf{d} + S(n)(T^n - T^{-n})\,\mathbf{e} \right] + S(n)\,\mathbf{d}. \tag{40}$$

The quadrature formula for S is

$$\int_0^n S(x)\,dx = \sum_{k=1}^{n-1} S(k) \left[1 - (2/(m+1)) \sum_{i=1}^{(m-1)/2} (\lambda_i^{n-k} + \lambda_i^{k-n})/(\lambda_i^n + \lambda_i^{-n}) \right]$$

$$+ S(0) \left[1/2 - (2/(m+1)) \right.$$

$$\left. \times \sum_{i=1}^{(m-1)/2} (\lambda_i^{2n} - \lambda_i)/\{(\lambda_i^{2n} + 1)(\lambda_i - 1)\} \right]$$

$$+ S(n) \left[1/2 - (2/(m+1)) \sum_{i=1}^{(m-1)/2} 1/(\lambda_i^n + \lambda_i^{-n}) \right]. \tag{41}$$

Proof. We shall sketch the main lines of the proof since it is similar to that of Theorems 1 and 6. Now, let $\mathbf{u}$ and $\mathbf{u}^*$ be even and odd alternating, respectively. Then, using $J\mathbf{u} = \mathbf{u}$ and $J\mathbf{u}^* = -\mathbf{u}^*$, (14) and (15) yield for an even–odd delta spline S_p at p (see the proof of Theorem 6):

$$\begin{aligned} \mathbf{u}_p^* &= 2(T^n + T^{-n})^{-1}(T^{-p} - T^p)\,\mathbf{e}, \qquad p \neq 0, n, \\ \mathbf{u}_p &= -2(T^n + T^{-n})^{-1}(T^{n-p} + T^{p-n})\,\mathbf{e}, \qquad p \neq 0, n, \end{aligned} \tag{42}$$

and

$$\begin{aligned} &\mathbf{u}_0 = -\mathbf{e} + (T^n + T^{-n})^{-1}(T^n - T^{-n})\,\mathbf{d}, \qquad \mathbf{u}_0^* = -2(T^n + T^{-n})^{-1}\,\mathbf{d}, \\ &\mathbf{u}_n = -2(T^n + T^{-n})^{-1}\,\mathbf{e}, \qquad \mathbf{u}_n^* = -(T^n + T^{-n})^{-1}(T^n - T^{-n})\,\mathbf{e} + \mathbf{d}. \end{aligned} \tag{43}$$

Substituting (42) and (43) into the relations

$$\mathbf{u} = \sum S(p)\,\mathbf{u}_p, \qquad \mathbf{u}^* = \sum S(p)\,\mathbf{u}_p^*,$$

we obtain (39) and (40).

Now, the quadrature formula (41) can be obtained by adding (39) and (40) and using the Euler–McLaurin formula (3). When this is done, we get ($\mathbf{b}^T\mathbf{u}^* = 0$, since $\mathbf{u}^*$ is odd alternating)

$$\begin{aligned} \int_0^n S(x)\,dx &= \sum_{k=1}^{n-1} S(k) + (S(0) + S(n))/2 - \mathbf{b}^T\mathbf{u} \\ &= \sum_{k=1}^{n-1} S(k)[1 + 2\mathbf{b}^T(T^n + T^{-n})^{-1}(T^{n-k} + T^{k-n})\,\mathbf{e}] \\ &\quad + S(0)[1/2 - \mathbf{b}^T(T^n + T^{-n})^{-1}(T^n - T^{-n})\,\mathbf{d} + \mathbf{b}^T\mathbf{e}] \\ &\quad + S(n)[1/2 + 2\mathbf{b}^T(T^n + T^{-n})^{-1}\,\mathbf{e}]. \end{aligned}$$

Using the relation $\mathbf{d} = (T+I)(T-I)^{-1}\,\mathbf{e}$ and Theorem 7, we obtain Eq. (41) in the same manner as used to obtain Eq. (4) of Theorem 2. This completes the proof of Theorem 9.

We now give an explicit formula for the null spaces of the operators $T^n + J$ and $T^n - J$ appearing in formulas (1) and (2).

THEOREM 10. *Let S be an mth order spline on $[0, n]$ with knots at the integers $0, 1, \ldots, n$, $n \geqslant 1$. Let $\bar{\mathbf{u}}$ and $\bar{\mathbf{u}}^*$ satisfy* (1) *and* (2). *Then there exist even alternating vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *such that*

$$\mathbf{u} = \bar{\mathbf{u}} + (T^n - I)^{-1}\,\mathbf{a} + (T^n + I)^{-1}\,\mathbf{b}, \tag{44}$$

$$\mathbf{u}^* = \bar{\mathbf{u}}^* + (T^n - I)^{-1}\,\mathbf{a} - (T^n + I)^{-1}\,\mathbf{b}, \tag{45}$$

and conversely, for any pair of even alternating vectors **a** *and* **b**, **u** *and* $\mathbf{u}^*$ *as given by* (44) *and* (45) *are solutions of* (1) *and* (2).

Proof. The verification of the fact that any **u** and $\mathbf{u}^*$ satisfying (44) and (45) also satisfy (1) and (2) is done by direct substitution and is omitted.

For the converse, let **u** and $\mathbf{u}^*$ satisfy (1) and (2) and define **a** and **b** by

$$\mathbf{a} = (T^n - I)[(\mathbf{u} + \mathbf{u}^*) - (\bar{\mathbf{u}} + \bar{\mathbf{u}}^*)],$$
$$\mathbf{b} = (T^n + I)[(\mathbf{u} - \mathbf{u}^*) - (\bar{\mathbf{u}} - \bar{\mathbf{u}}^*)].$$

TABLE I

Weights $W_{k,m,n}$ of the Quadrature Formulas for the mth Order Even Alternating Spline on $[0, n]$

$n = 2$

m	$k = 0$ and 2	$k = 1$
3	0.37500000	1.2500000
5	0.36458333	1.2708333
7	0.36351103	1.2729779
9	0.36339465	1.2732107
11	0.36338182	1.2732363
13	0.36338040	1.2732392

$n = 3$

m	$k = 0$ and 3	$k = 1$ and 2
3	0.40000000	1.10000000
5	0.39743590	1.10256410
7	0.39734577	1.10265423
9	0.39734235	1.10265765
11	0.39734222	1.10265778
13	0.39734221	1.10265779

$n = 5$

m	$k = 0$ and 5	$k = 1$ and 4	$k = 2$ and 3
3	0.39473684	1.13157895	0.97368421
5	0.38789436	1.14841617	0.96368945
7	0.38676028	1.151373825	0.96186590
9	0.38655478	1.151911686	0.96153353
11	0.38651715	1.152010202	0.96147264
13	0.38651025	1.152028283	0.96141647

From this it is readily seen that $\mathbf{u}$ and $\mathbf{u}^*$ satisfy (44) and (45), so it remains to show that $\mathbf{a}$ and $\mathbf{b}$ are even alternating, that is, $J\mathbf{a}=\mathbf{a}$ and $J\mathbf{b}=\mathbf{b}$. Now,

$$J\mathbf{a}=(T^{-n}-I)\,J[(\mathbf{u}+\mathbf{u}^*)-(\bar{\mathbf{u}}+\bar{\mathbf{u}}^*)] \tag{46}$$

and

$$J\mathbf{b}=(T^{-n}+I)\,J(\mathbf{u}-\mathbf{u}^*)-(\bar{\mathbf{u}}-\bar{\mathbf{u}}^*)]. \tag{47}$$

Since $(\mathbf{u}+\mathbf{u}^*)$ and $(\bar{\mathbf{u}}+\bar{\mathbf{u}}^*)$ both satisfy (1) we also have

$$(T^n+J)[(\mathbf{u}+\mathbf{u}^*)-(\bar{\mathbf{u}}+\bar{\mathbf{u}}^*)]=0,$$

and similarly from (2) we get

$$(T^n-J)[(\mathbf{u}-\mathbf{u}^*)-(\bar{\mathbf{u}}-\bar{\mathbf{u}}^*)]=0.$$

So,

$$J[(\mathbf{u}+\mathbf{u}^*)-(\bar{\mathbf{u}}+\bar{\mathbf{u}}^*)]=-T^n[(\mathbf{u}+\mathbf{u}^*)-(\bar{\mathbf{u}}-\bar{\mathbf{u}}^*)]$$

and

$$J[(\mathbf{u}-\mathbf{u}^*)-(\bar{\mathbf{u}}-\bar{\mathbf{u}}^*)]=T^n[(\mathbf{u}-\mathbf{u}^*)-(\bar{\mathbf{u}}-\bar{\mathbf{u}}^*)].$$

Using these relations in (46) and (47) we see that $J\mathbf{a}=-\mathbf{a}$ and $J\mathbf{b}=-\mathbf{b}$, that is, $\mathbf{a}$ and $\mathbf{b}$ are even alternating. This completes the proof of Theorem 10.

We now present Tables I and II, which give computations of the weights in the quadrature formulas for even alternating splines. These weights are

TABLE II

Weights $W_{k,m}$ of the Quadrature Formulas for the mth Order Even Alternating Semicardinal Spline on $[0,\infty)$

k \ m	3	7	9	13
1	0.39433757	0.38327658	0.38209248	0.38107220
2	1.13397460	1.16674594	1.17101836	1.17485423
3	0.96410162	0.92459288	0.91755264	0.91073966
4	1.00961894	1.03880320	1.04660910	1.05507353
5	0.99742261	0.97941944	0.97233813	0.96351465
6	1.0069061	1.01099310	1.01668334	1.02492993
7	0.99981495	0.99411846	0.98988375	0.98272207
8	1.00004958	1.00314892	1.00614518	1.01205473
9	0.99998671	0.99831502	0.99626485	0.99156291
10	1.00000356	1.00090193	1.00227074	1.00591395
11	0.99999905	0.99951721	0.99861944	0.99585170
12	1.00000026	1.00025843	1.00083937	1.00291078

explicitly given by Theorems 2 and 4. In these tables, $W_{k,m,n}$ is the kth weight, $0 \leqslant k \leqslant n$, in the quadrature formula of the mth order even alternating spline on $[0, n]$. These weights are very simple to compute and the tables are provided for illustrative purposes only. The tables clearly show convergence, as expected, to the semicardinal case as $n \to \infty$. Moreover, based on computations of the weights as well as analytical considerations, we also believe that the weights converge as $m \to \infty$ to those obtained by integrating the function $f_k(x) = a + bx + \sum_{i=1}^{n-1} c_i \sin(i\pi x/n)$, with a, b and c_i chosen that $f_k(j) = \delta_{jk}$. If this conjecture holds, it would serve to distinguish the present formulas for even alternating splines from the semicardinal formulas of Schoenberg and Silliman [4], derived from natural splines. The values of the zeroes of the Euler–Frobenius polynomials that were used in computing the tables of weights are those given in Schoenberg and Silliman [4].

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